

Contents

Synopsis	xi
Preface	xvi
Part I. Operators with Index and Homotopy Theory	1
Chapter 1. Fredholm Operators	2
1. Hierarchy of Mathematical Objects	2
2. The Concept of Fredholm Operator	3
3. Algebraic Properties. Operators of Finite Rank. The Snake Lemma	5
4. Operators of Finite Rank and the Fredholm Integral Equation	9
5. The Spectra of Bounded Linear Operators: Basic Concepts	10
Chapter 2. Analytic Methods. Compact Operators	12
1. Analytic Methods. The Adjoint Operator	12
2. Compact Operators	18
3. The Classical Integral Operators	25
4. The Fredholm Alternative and the Riesz Lemma	26
5. Sturm-Liouville Boundary Value Problems	28
6. Unbounded Operators	34
7. Trace Class and Hilbert-Schmidt Operators	53
Chapter 3. Fredholm Operator Topology	63
1. The Calkin Algebra	63
2. Perturbation Theory	65
3. Homotopy Invariance of the Index	68
4. Homotopies of Operator-Valued Functions	72
5. The Theorem of KUIPER	77
6. The Topology of $\mathcal{F}$	81
7. The Construction of Index Bundles	82
8. The Theorem of Atiyah-Jänich	88
9. Determinant Line Bundles	91
10. Essential Unitary Equivalence and Spectral Invariants	108
Chapter 4. Wiener-Hopf Operators	120
1. The Reservoir of Examples of Fredholm Operators	120
2. Origin and Fundamental Significance of Wiener-Hopf Operators	121
3. The Characteristic Curve of a Wiener-Hopf Operator	122
4. Wiener-Hopf Operators and Harmonic Analysis	123
5. The Discrete Index Formula. The Case of Systems	125
6. The Continuous Analogue	129

Part II. Analysis on Manifolds	133
Chapter 5. Partial Differential Equations in Euclidean Space	134
1. Linear Partial Differential Equations	134
2. Elliptic Differential Equations	137
3. Where Do Elliptic Differential Operators Arise?	139
4. Boundary-Value Conditions	141
5. Main Problems of Analysis and the Index Problem	143
6. Numerical Aspects	143
7. Elementary Examples	144
Chapter 6. Differential Operators over Manifolds	156
1. Differentiable Manifolds — Foundations	157
2. Geometry of C^∞ Mappings	160
3. Integration on Manifolds	166
4. Exterior Differential Forms and Exterior Differentiation	171
5. Covariant Differentiation, Connections and Parallelity	176
6. Differential Operators on Manifolds and Symbols	181
7. Manifolds with Boundary	190
Chapter 7. Sobolev Spaces (Crash Course)	193
1. Motivation	193
2. Definition	194
3. The Main Theorems on Sobolev Spaces	200
4. Case Studies	203
Chapter 8. Pseudo-Differential Operators	206
1. Motivation	206
2. Canonical Pseudo-Differential Operators	210
3. Principally Classical Pseudo-Differential Operators	214
4. Algebraic Properties and Symbolic Calculus	228
5. Normal (Global) Amplitudes	231
Chapter 9. Elliptic Operators over Closed Manifolds	237
1. Mapping Properties of Pseudo-Differential Operators	237
2. Elliptic Operators — Regularity and Fredholm Property	239
3. Topological Closure and Product Manifolds	242
4. The Topological Meaning of the Principal Symbol — A Simple Case Involving Local Boundary Conditions	244
Part III. The Atiyah-Singer Index Formula	251
Chapter 10. Introduction to Topological K -Theory	252
1. Winding Numbers	252
2. The Topology of the General Linear Group	257
3. Elementary K -Theory	262
4. K -Theory with Compact Support	266
5. Proof of the Periodicity Theorem of R. BOTT	269
Chapter 11. The Index Formula in the Euclidean Case	275
1. Index Formula and Bott Periodicity	275

2. The Difference Bundle of an Elliptic Operator	276
3. The Index Theorem for $\text{Ell}_c(\mathbb{R}^n)$	281
Chapter 12. The Index Theorem for Closed Manifolds	284
1. Pilot Study: The Index Formula for Trivial Embeddings	285
2. Proof of the Index Theorem for Nontrivial Normal Bundle	287
3. Comparison of the Proofs	301
Chapter 13. Classical Applications (Survey)	310
1. Cohomological Formulation of the Index Formula	311
2. The Case of Systems (Trivial Bundles)	316
3. Examples of Vanishing Index	317
4. Euler Characteristic and Signature	319
5. Vector Fields on Manifolds	325
6. Abelian Integrals and Riemann Surfaces	329
7. The Theorem of Hirzebruch-Riemann-Roch	333
8. The Index of Elliptic Boundary-Value Problems	337
9. Real Operators	356
10. The Lefschetz Fixed-Point Formula	357
11. Analysis on Symmetric Spaces: The G -equivariant Index Theorem	360
12. Further Applications	362
Part IV. Index Theory in Physics and the Local Index Theorem	363
Chapter 14. Physical Motivation and Overview	364
1. Classical Field Theory	365
2. Quantum Theory	373
Chapter 15. Geometric Preliminaries	394
1. Principal G -Bundles	394
2. Connections and Curvature	396
3. Equivariant Forms and Associated Bundles	400
4. Gauge Transformations	409
5. Curvature in Riemannian Geometry	414
6. Bochner-Weitzenböck Formulas	434
7. Characteristic Classes and Curvature Forms	443
8. Holonomy	455
Chapter 16. Gauge Theoretic Instantons	460
1. The Yang-Mills Functional	460
2. Instantons on Euclidean 4-Space	466
3. Linearization of the Moduli Space of Self-dual Connections	489
4. Manifold Structure for Moduli of Self-dual Connections	496
Chapter 17. The Local Index Theorem for Twisted Dirac Operators	513
1. Clifford Algebras and Spinors	513
2. Spin Structures and Twisted Dirac Operators	525
3. The Spinorial Heat Kernel	538
4. The Asymptotic Formula for the Heat Kernel	549
5. The Local Index Formula	576

6. The Index Theorem for Standard Geometric Operators	594
Chapter 18. Seiberg-Witten Theory	643
1. Background and Survey	643
2. Spin^c Structures and the Seiberg-Witten Equations	655
3. Generic Regularity of the Moduli Spaces	663
4. Compactness of Moduli Spaces and the Definition of S-W Invariants	685
Appendix A. Fourier Series and Integrals - Fundamental Principles	705
1. Fourier Series	705
2. The Fourier Integral	707
Appendix B. Vector Bundles	712
1. Basic Definitions and First Examples	712
2. Homotopy Equivalence and Isomorphy	716
3. Clutching Construction and Suspension	718
Bibliography	723
Index of Notation	741
Index of Names/Authors	749
Subject Index	757