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**Semi-Analytical Finite Element  
Formulations for Layered Com-  
posite Shells with Consideration  
of Edge Effects**

Reihe **18**: Mechanik/  
Bruchmechanik

Nr. **203**

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