CONTENTS

Preface ix

Acknowledgements xi

PART I GENERAL INTRODUCTION 1

PART II ON THE NAVIER–STOKES EQUATIONS 15

1 Some elements of functional analysis 17
- 1.1 Function spaces 17
- 1.2 The Stokes problem 19
- 1.3 A brief overview of Sobolev spaces 22
  - 1.3.1 Definition in the case of the whole space $\mathbf{R}^d$ 22
  - 1.3.2 Sobolev embeddings 23
  - 1.3.3 A compactness result 27
- 1.4 Proof of the regularity of the pressure 28

2 Weak solutions of the Navier–Stokes equations 33
- 2.1 Spectral properties of the Stokes operator 33
  - 2.1.1 The case of bounded domains 33
  - 2.1.2 The general case 36
- 2.2 The Leray theorem 42
  - 2.2.1 Construction of approximate solutions 44
  - 2.2.2 A priori bounds 46
  - 2.2.3 Compactness properties 47
  - 2.2.4 End of the proof of the Leray theorem 48

3 Stability of Navier–Stokes equations 53
- 3.1 The time-dependent Stokes problem 53
- 3.2 Stability in two dimensions 56
- 3.3 Stability in three dimensions 58
- 3.4 Stable solutions in a bounded domain 64
  - 3.4.1 Intermediate spaces 64
  - 3.4.2 The well-posedness result 66
  - 3.4.3 Some remarks about stable solutions 71
- 3.5 Stable solutions in a domain without boundary 72
- 3.6 Blow-up condition and propagation of regularity 77
  - 3.6.1 Blow-up condition 77
  - 3.6.2 Propagation of regularity 80

4 References and remarks on the Navier–Stokes equations 83

PART III ROTATING FLUIDS 85

5 Dispersive cases 87
5.1 A brief overview of dispersive phenomena 87
5.1.1 Strichartz-type estimates 89
5.1.2 Illustration of the wave equation 91
5.2 The particular case of the Rossby operator in $\mathbf{R}^3$ 93
5.3 Application to rotating fluids in $\mathbf{R}^3$ 100
5.3.1 Study of the limit system 101
5.3.2 Existence and convergence of solutions to the rotating-fluid equations 102
5.3.3 Global well-posedness 108

6 The periodic case 117
6.1 Setting of the problem, and statement of the main result 117
6.2 Derivation of the limit system in the energy space 121
6.3 Properties of the limit quadratic form Q 125
6.4 Global existence and stability for the limit system 132
6.5 Construction of an approximate solution 140
6.6 Study of the limit system with anisotropic viscosity 144

7 Ekman boundary layers for rotating fluids 155
7.1 The well-prepared linear problem 159
7.2 Non-linear estimates in the well-prepared case 168
7.3 The convergence theorem in the well-prepared case 172
7.4 The ill-prepared linear problem 176
7.5 Non-linear estimates in the ill-prepared case 199
7.6 The convergence theorem in the whole space 202
7.7 The convergence theorem in the periodic case 210
7.7.1 Proof of the theorem 210

8 References and remarks on rotating fluids 217

PART IV PERSPECTIVES 219

9 Stability of horizontal boundary layers 221
9.1 Critical Reynolds number 221
9.2 Energy of a small perturbation 224
9.3 Rolls and turbulence 225

10 Other systems 227
10.1 Large magnetic fields 227
10.2 A rotating MHD system 228
10.3 Quasigeostrophic limit 228

11 Vertical layers 231
11.1 Introduction 231
11.2 $E^{1/3}$ layer 232

11.3 $E^{1/4}$ layer 232
11.4 Mathematical problems 235

12 Other layers 237
12.1 Sphere 237
12.2 Spherical shell 237
12.3 Layer between two differentially rotating spheres 238

References 239

List of Notations 247

Index 249