

CONTENTS.

CHAPTER I.

COORDINATES.

	PAGE
Descriptive and metrical theorems	1
General definition of trilinear coordinates	2
Relation between point (1, 1, 1) and line $x + y + z = 0$	4
Particular case of trilinear coordinates	5
circular coordinates	6
Circular points at infinity	7
LINE COORDINATES	9
Their relation to trilinear coordinates	10
Particular cases of line-coordinates	11
Geometrical duality	12

CHAPTER II.

GENERAL PROPERTIES OF ALGEBRAIC CURVES.

SECTION I. NUMBER OF TERMS IN THE EQUATION	13
All forms which are general must have as many independent constants as the general equation	13
Number of terms in the general equation	15
Number of points which determine a n -ic	15
A single curve determined by these conditions	16
In what cases this number of points fails to determine a n -ic	17
If one less than this number of points be given, the curve passes through other fixed points	18
If of intersections of two n -ic's, np lie on a p -ic, the remainder lie on a $(n - p)$ -ic	18
Extension of Pascal's theorem	19

	PAGE
Steiner's and Kirkman's theorems on the hexagon	19
Theorems concerning the intersections of two curves	20
SECTION II. MULTIPLE POINTS AND TANGENTS	22
Equation of tangent at origin	23
The origin, a double point	24
Three kinds of double points	24
Their relation illustrated	26
Triple points, their species	27
Number of nodes equivalent to a multiple point	28
Multiple point equivalent to how many conditions	29
Limit to number of nodes on a proper curve	29
Deficiency of a curve	30
Fundamental property of unicursal curves	31
Consideration of the case where the axis is a multiple tangent	32
Stationary tangents and inflexions	34
Two consecutive tangents coincide at a stationary tangent	34
Correspondence of reciprocal singularities	34
Curve crosses the tangent at an inflexion	35
Measure of inclination of curve to the axis	36
Points of undulation	37
Relation between points where tangents meet the curve again	38
Equation of asymptotes, how formed	39
Examples	41
SECTION III. TRACING OF CURVES	42
Newton's process for determining form of curve at a singular point	46
Keratoid and ramphoid cusps	48
SECTION IV. POLES AND POLARS	49
Joachimsthal's method of determining points where a line meets a curve	49
Polar curves	50
of origin	51
Every right line has $(n - 1)^2$ poles	51
Multiple points and cusps, how related to polar curves	52
SECTION V. GENERAL THEORY OF MULTIPLE POINTS AND TANGENTS	52
All polars of point on curve touch at that point	53
Points of contact of tangents from a point how determined	53
Degree of reciprocal of a curve	54
Effect of singularities on degree of reciprocal	55
Discriminant of a curve	55
If the first polar of A has a double point B , polar conic of B has a double point A	56
Hessian and Steinerian defined	57
Conditions for a cusp	58
Number of points of inflexion	59
how affected by multiple points	60
Equation of system of tangents from a point	61
Application to the case of a cubic	62
Number of tangents from a multiple point	63
SECTION VI. RECIPROCAL CURVES	63
What singularities to be counted ordinary	64
Plücker's equations	65
Number of conditions, the same for curve and its reciprocal	66
Deficiency, the same for both	66
Cayley's modification of Plücker's equations	66

CHAPTER III.

ENVELOPES.

	PAGE
Two forms in which problem of envelopes presents itself	67
Envelope of curve whose equation contains a single parameter	68
Envelope of $a \cos^2 \theta + b \sin^2 \theta = c$	69
Equation of parallel to a conic	70
Envelope of right line containing a single parameter algebraically: its characteristics	70
Envelope of curve with related parameters	72
Method of indeterminate multipliers	73
Envelope of curve whose equation contains independent parameters	74
Explanation of difficulty in theory of envelopes	74
RECIPROCAL CURVES	76
Method of finding equation of reciprocal	77
Reciprocal of a cubic	77
Symbolical form of equation of reciprocal	78
Reciprocal of a quartic	78
Equation of system of tangents from a point	79
Equation of reciprocal in polar coordinates	79
TAUT-INVARIANT of two curves	80
Its order in the coefficients of each curve	81
EVOLUTES	82
Defined as envelope of normals	83
Coordinates of centre of curvature	84
General expression for radius of curvature	87
Length of arc of evolute	88
Radius of curvature in polar coordinates	88
Evolutes of curves given by tangential equation	89
Quasi-normals and quasi-evolutes	90
Quasi-evolute of conic	91
General form of equation of quasi-normal	92
Quasi-evolute when the absolute is a conic	93
Normal of a point at infinity	94
Characteristics of evolute	94
Deficiency of evolute	97
Condition that four consecutive points on a curve should be concircular	97
CAUSTICS	98
Caustic by reflection of a circle	98
Queletelet's method	98
Pedal of a curve	99
Caustic by refraction of right line and circle	100
Evolute of Cartesian	101
PARALLEL CURVES AND NEGATIVE PEDALS	101
Cayley's formulae for characteristics of parallel curves	102
Problem of negative pedals	105
Roberts's method	105
Method of inversion	106
Characteristics of inverse curve and of pedal	106
Caustic by reflexion of parabola	107
Negative pedal of central and focal ellipse	107

CHAPTER IV.

METRICAL PROPERTIES.

	PAGE
Newton's theorem of constant ratio of rectangles	108
Carnot's theorem of transversals	109
Three inflexions of cubics lie on a right line	110
Two kinds of bitangents of quartic	111
DIAMETERS	112
Newton's generalization of notion of diameters	112
Theorem concerning intercept made between curves and asymptotes	113
Curvilinear diameters	113
Centres	115
POLES AND POLARS	115
Cotes's theorem of harmonic means of radii	115
Polar curves	116
Polars of points at infinity	117
Pole of line at infinity	117
MacLaurin's extension of Newton's theorem	117
Pole of line at infinity with respect to curve of n^{th} class	118
Its metrical property : centre of mean distances of contacts of parallel tangents	119
FOCI	119
General definition of foci	119
Number of foci possessed by a curve	120
Antipoints	122
Coordinates of foci how found	122
Locus of a point the tangents from which make with a fixed line angles whose sum is constant	123
Every focus of a curve is a focus of its evolute	124
Theorems concerning focal perpendiculars on tangents	124
concerning angles between focal radii and tangent	125
concerning focal distances of point on curve	126
Locus of double focus of circular curve determined by $N - 3$ points	127
Locus of focus of curve of n^{th} class determined by $N - 1$ tangents	127
Miquel's theorem as to foci of parabolas touching 4 of 5 given lines	128

CHAPTER V.

CUBICS.

General division of cubics	129
SECTION I. INTERSECTION OF A CUBIC WITH OTHER CURVES	130
Tangential of a point and satellite of a line defined	130
Asymptotes meet curve in three collinear finite points	131
Three points of inflexion lie on a right line	131
Four points of contact of tangents from any point on curve, how related	132
MacLaurin's theory of correspondence of points on a cubic	133
Coresidual of four points on a cubic	134
To draw a conic having four-point contact and elsewhere touching a cubic	135
Conic of 5-point contact, how constructed	135
Sextactic points on cubic, how found	135
Sylvester's theory of residuation	136

	PAGE
Two coresidual points must coincide	137
Two systems coresidual to the same are coresidual to each other	138
Analogues in theory of cubics to anharmonic theorems of conics	140
Locus of common vertex of two triangles whose bases are given and vertical angles equal, or having a given difference	142
SECTION II. POLES AND POLARS.	142
Construction for polar of a point with respect to a triangle	143
Construction by the ruler for polar of a point with respect to a cubic	143
Anharmonic ratio constant of pencil of four tangents from any point on a cubic	144
Two classes of non-singular cubics	145
Sixteen foci of a circular cubic lie on four circles	145
Chords through a point on cubic cut harmonically by polar conic	146
Harmonic polar of point of inflexion	146
All cubics through nine points of inflexion have these for inflexions	148
Correspondence of two points on Hessian	149
Steinerian of a cubic identical with its Hessian	150
Cayleyan, different definitions of	151
Polar line with respect to cubic of point on Hessian touches Hessian	152
Common tangents of cubic and Hessian	152
Stationary tangents touch the Hessian	152
Tangents to Hessian at corresponding points meet on Hessian	153
Three cubics have common Hessian	153
Rule for finding point of contact of any tangent to Cayleyan	154
Points of contact of stationary tangents with Cayleyan	155
Coordinates of tangential of point on cubic, how found	156
Polar conic of line with respect to cubic	156
How related to triangle formed by tangents where line meets cubic	157
Double points, how situated with regard to polar conics of lines	157
Polar conic of line infinity	158
Another method of obtaining tangential equation of cubic	158
Polar conic of a line when reduces to a point	158
Points, whose polar with respect to two cubics are the same	159
Critic centres of system of cubics	160
Locus of nodes of nodal cubics through seven points	160
Plücker's classification of cubics	161
SECTION III. CLASSIFICATION OF CUBICS	162
Every cubic may be projected into one of five divergent parabolas	164
and into one of five central cubics	164
Classification of cubic cones	165
No real tangents can be drawn from oval	167
Unipartite and bipartite cubics	168
Species of cubics	169
Newton's method of reducing the general equation	177
Plücker's groups	178
SECTION IV. UNICURSAL CUBICS	179
Inscription of polygons in unicursal cubics	181
Cissoid, its properties	182
Acnodal cubic has real inflexions, crunodal imaginary	184
Construction for acnode given three inflexional tangents	184
General expression for coordinates in terms of parameter	185
SECTION V. INVARIANTS AND COVARIANTS OF CUBICS	188
Canonical form of cubic	188

	PAGE
Notation for general equation	189
General equation of Hessian and of Cayleyan	190
Invariant S , and its symbolical form	191
Invariant T	192
General equation of reciprocal	193
Calculation of invariants by the differential equation	194
Discriminant expressed in terms of fundamental invariants	196
Hessian of $\lambda U + \mu H$	196
Conditions that general equation should represent three right lines	197
Reduction of general equation to canonical form	198
Expression of discriminant in terms of fundamental invariants	199
Of anharmonic ratio of four tangents from any point on curve	199
Covariant cubics expressed in form $\lambda U + \mu H$	201
Sextic covariants	202
The skew covariant	202
Equation of nine inflexional tangents	203
Equation of Cayleyan in point coordinates	203
Identical equation in theory of cubics	205
Conic through five consecutive points on cubic	207
Equation expressed in four line coordinates	209
Conditions that cubic should represent conic and line	210
Discriminant of cubic expressed as determinant	211
Hessian of PU and UV	212

CHAPTER VI.

QUARTICS.

Genera of quartics	213
Special forms of quartics	214
Illustration of the different forms	215
Distinction of real and imaginary	217
Flecnodes and biflecnodes	217
Quartic may have four real points of undulation	218
Quartics may be quadripartite	219
Zeuthen's classification of quartics	220
Number of real bitangents	220
Inflexions of quartics, how many real	221
Classification of quartics in respect of their infinite branches	222
THE BITANGENTS	223
Discussion of equation $UW = V^2$	224
There are 315 conics passing through eight contacts of bitangents	227
Scheme of these conics	228
Hesse's algorithm for the bitangents	230
Geiser's method of connecting bitangents with solid geometry	231
Cayley's rule of bifid substitution	232
Bitangents whose contacts lie on a cubic	234
Aronhold's discussion of the bitangents	234
From 7 bitangents the rest can be found by linear constructions	237
Aronhold's algebraic investigation	238
BINODAL AND BICIRCULAR QUARTICS	240
Tangents from nodes of a binodal are homographic	241
Foci of bicircular quartic lie on four circles	242

	PAGE
Casey's generation of bicircular quartics	243
Two classes of bicircular quartics	246
Relations connecting focal distances of point on bicircular	246
Confocal bicirculars cut at right angles	247
Hart's investigation	248
Cartesians	250
The limaçon and the cardioide	252
Focal properties obtained by inversion	252
Inscription of polygons in binodal quartics	253
UNICURSAL QUARTICS	254
Correspondence between conics and trinodal quartics	254
Tangents at or from nodes touch the same conic	256
Tacnodal and oscnodal quartics	258
Triple points	259
Expression of coordinates by a parameter	260
INVARIANTS AND COVARIANTS OF QUARTICS	263
General quartic cannot be reduced to sum of five fourth powers	265
Covariant quartics	269
Examination of special case	269
Covariant conics	273

CHAPTER VII.

TRANSCENDENTAL CURVES.

The cycloid	275
Geometric investigation of its properties	276
Epicycloids and epitrochoids	278
Their evolutes are similar curves	281
Examples of special cases	282
Limaçon generated as epicycloid	282
Steiner's envelope	283
Reciprocal of epicycloid	283
Radius of curvature of roulettes	284
Trigonometric curves	285
Logarithmic curves	286
Catenary	287
Tractrix and syntractrix	288
Curves of pursuit	290
Involute of circle	290
Spirals	291

CHAPTER VIII.

TRANSFORMATION OF CURVES.

LINEAR TRANSFORMATION	295
Anharmonic ratio unaltered by linear transformation	296
Three points unaltered by linear transformation	297
Projective transformation	298

Homographic transformation may be reduced to projection	PAGE 299
INTERCHANGE OF LINE AND POINT COORDINATES	301
Method of skew reciprocals	303
Skew reciprocals reducible to ordinary reciprocals	306
QUADRIC TRANSFORMATIONS	308
Inversion, a case of quadric transformation	310
Applications of method of inversion	312
RATIONAL TRANSFORMATION	313
Roberts's transformation	313
Cremona's rational transformation	316
If three curves have common point their Jacobian passes through it	319
Deficiency unaltered by Cremona transformation	321
Every Cremona transformation may be reduced to a succession of quadric transformations	322
TRANSFORMATION OF A GIVEN CURVE	324
Rational transformation between two curves	324
Deficiency unaltered by rational transformation	326
Transformation, so that the order of the transformed curve may be as low as possible	329
Expression of coordinates by means of elliptic functions when $D = 1$	330
and by means of hyper-elliptic functions when $D = 2$	330
Theorem of constant deficiency derived from theory of elimination	331
CORRESPONDENCE OF POINTS ON A CURVE	331
Collinear correspondence of points	331
Correspondence on a unicursal curve	332
Number of united points	333
Correspondence on curves in general	334
Inscription of polygons in conics	337
in cubics	337

CHAPTER IX.

GENERAL THEORY OF CURVES.

Cayley's method of solving the general problem of bitangents	341
Order of bitangential curve	342
Hesse's reduction of bitangential equation	344
Bitangential of a quartic	349
Second method of solving problem of double tangents	351
Formation of equation of tangential curve	355
Application to quartic	356
POLES AND POLARS	357
Jacobian, properties of	357
Steiner's theorems on systems of curves	360
Tact-invariants	360
Discriminant of discriminant of $\lambda u + \mu v$ and of $\lambda u + \mu v + \nu w$	362
Condition for point of undulation	362
For coincidence of double and stationary tangent	362
Steinerian of a curve	363
Its characteristics	363
The Cayleyan or Steiner-Hessian	364
Its characteristics	365

CONTENTS.

xix

	PAGE
Generalization of the theory	365
OSCULATING CONICS	368
Aberrancy of curvature	368
Investigation of conic of 5-point contact	370
Determination of number of sextactic points	372
SYSTEMS OF CURVES	372
Chasles' method	373
Characteristics of systems of conics	374
Number of conics which touch five given curves	375
Zeuthen's method	377
Degenerate curves	377
Cayley's table of results	380
Number of conics satisfying five conditions of contact with other curves	382
Professor Cayley's note on degenerate forms of curves	383

NOTES.

Professor Cayley on the bitangents of a quartic	387
---	-----