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# Numerical Methods for Metric Graphs

Eigenvalue Problems and Parabolic Partial  
Differential Equations

 Springer

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