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Topological Methods in
the Theory of Nonlinear
Integral Equations

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CONTENTS

FOREWORD	xi
INTRODUCTION	1
I. Nonlinear operators	11
1. Fundamental concepts	12
<i>Operators in Banach Spaces.</i> (12) <i>The spaces C and L^p.</i> (14)	
<i>Completely continuous operators.</i> (15) <i>Linear integral operators.</i> (18)	
2. The operator f	20
<i>Convergence in measure.</i> (20) <i>Continuity of the operator f.</i> (22)	
<i>Boundedness of the operator f.</i> (26) <i>Sufficient conditions for the continuity of the operator f.</i> (27) <i>The space C</i> (32)	
3. Nonlinear integral operators	32
<i>P. S. Uryson's operator.</i> (32) <i>P. S. Uryson's operator in the space C.</i> (33) <i>Auxiliary lemmas.</i> (34) <i>P. S. Uryson's operator in the space L^p.</i> (38) <i>Hammerstein's operator.</i> (46) <i>A. M. Lyapunov's integral power series.</i> (47)	
4. Splitting of linear operators	48
<i>Inequality for moments.</i> (49) <i>Splitting of a linear operator acting in L^2.</i> (50) <i>Splitting of an operator, acting from L^q to L^p.</i> (51) <i>Integral operators with kernels, the iterates of which are P_p-th power summable.</i> (60) <i>Kernels, certain iterates of which are bounded.</i> (64) <i>Kernels with eigenvalues of different signs</i> (66)	
5. Weakly continuous functionals	67
<i>Differentiable functionals.</i> (67) <i>Weakly continuous functionals</i> <i>E. S. Tsilnadze's theorems</i> (71)	
II. The rotation of a vector field	77
1. Vector field in a finite-dimensional space	79
<i>The degree of a mapping.</i> (79) <i>Numerations.</i> (80) <i>The degree</i>	

	<i>of a numeration.</i> (82) <i>The rotation of a vector field.</i> (85) <i>Homotopic vector fields.</i> (87) <i>Fixed points.</i> (89) <i>Lemma on the product of rotations</i> (91)	
2.	The rotation of concrete vector fields.	91
	<i>The "hedgehog" theorem.</i> (91) <i>Linear vector fields.</i> (93) <i>Symmetrical numerations on the sphere.</i> (94) <i>The theorem of L. A. Lyusternik, L. G. Shnirel'man and K. Borsuk.</i> (98) <i>Closed coverings of a sphere</i> (100)	
3.	Completely continuous vector fields.	103
	<i>Finite-dimensional lemma.</i> (103) <i>The rotation of a completely continuous vector field.</i> (105) <i>Homotopic fields.</i> (108) <i>The algebraic number of fixed points of a completely continuous vector field.</i> (109) <i>Classification theorem.</i> (110) <i>Correctness of formulation of the problem.</i> (112) <i>Extension of a completely continuous vector field.</i> (113) <i>The possibility of extending a vector field with no null vector.</i> (118) <i>The rotation of a field on the boundary of a bounded region</i> (121)	
4.	The Leray-Schauder principle	123
	<i>The general fixed points principle.</i> (123) <i>Schauder's Principle.</i> (124) <i>Odd fields on the sphere.</i> (124) <i>Almost linear vector fields.</i> (126) <i>Vector fields which are symmetric with respect to a subspace.</i> (127) <i>The product of the rotations of completely continuous vector fields.</i> (129) <i>Linear fields.</i> (133) <i>Calculation of the index of a fixed point</i> (135)	
III. Existence theorems		141
1.	The contraction mapping principle	141
	<i>The contraction mapping principle.</i> (141) <i>The resolvent of a nonlinear operator and its properties.</i> (143) <i>Existence theorems.</i> (146) <i>Approximate calculation of the eigenvalues and eigenfunctions of a perturbed linear operator</i> (151)	
2.	Application of topological fixed point principles to the proof of existence theorems.	157
	<i>General formulation of the problem.</i> (157) <i>Vector fields which are not completely continuous.</i> (159) <i>Odd integral power series.</i> (160) <i>Almost linear equations. Local theorems.</i> (163) <i>Almost linear equations. Non-local theorems</i> (165)	
3.	Convergence of B. G. Galerkin's method for the approximate solution of nonlinear equations	169
	<i>Convergence of the process.</i> (170) <i>Speed of convergence.</i> (172) <i>B. G. Galerkin's method</i> (174) <i>G. I. Petrov's generalization.</i> (175) <i>Calculation of the eigenvalues of linear operators</i> (178)	

IV. Problems concerning eigenfunctions	181
1. Topological principles of the existence of an eigenvector. 184 <i>Approximate eigenvectors.</i> (184) <i>The principle of comparison for two completely continuous vector fields.</i> (186) <i>Eigenfunctions non-zero index.</i> (187) <i>Small eigenfunctions. The method of small perturbations</i> (189)	
2. The justification of linearisation in the problem of bifurcation points 191 <i>Bifurcation points.</i> (191) <i>Continuous branches of eigenvectors.</i> (194) <i>The case of a completely continuous operator.</i> (195) <i>Examples.</i> (196) <i>The structure of the spectrum.</i> (198) <i>The general case.</i> (200) <i>A. M. Lyapunov's operator</i> (206)	
3. Asymptotically linear operators 206 <i>Asymptotic bifurcation points.</i> (206) <i>Criterion for continuity of the spectrum</i> (210)	
4. Theorems of Lyapunov type 211 <i>The index of a fixed point.</i> (211) <i>The case of a real space.</i> (223) <i>The case of a complex space.</i> (228) <i>Branching points</i> (229)	
5. The distribution of the spectrum in the neighbourhood of of a bifurcation point. 232 <i>The sets $N_{\varepsilon, \delta}^-$ and $N_{\varepsilon, \delta}^+$.</i> (232) <i>A theorem on bifurcation points.</i> (232) <i>A differentiable operator.</i> (234) <i>Examples</i> (236)	
V. Eigenfunctions of positive operators	239
1. A theorem on the eigenvector 240 <i>The cone in Banach space.</i> (240) <i>Positive operators.</i> (242) <i>Branches of eigenvectors.</i> (248) <i>The cone $K_{u_0 k}$</i> (250)	
2. Operators with monotonic minorants..... 254 <i>The construction of operators near to the one under study.</i> (254) <i>Monotonic operators.</i> (255) <i>Homogeneous operators.</i> (257) <i>Linear operators.</i> (259) <i>Linear u_0-bounded operators.</i> 261) <i>Monotonic minorants</i> (267)	
3. Uryson's theorems on the spectrum 272 <i>The set of eigenvectors is closed.</i> (272) <i>The positive spectrum fills an interval.</i> (274) <i>The bounds of a positive spectrum.</i> (277) <i>Nonlinear u_0-concave operators.</i> (281) <i>u_0-monotonic operators.</i> (286) <i>Continuous dependence on a parameter.</i> (288) <i>P. S. Uryson's theorems</i> (289)	

VI. Variational methods	299
1. Existence theorems	300
<i>General principle.</i> (300) <i>Existence theorems for Hammerstein equations with positive definite kernels.</i> (304) <i>Existence theorems for Hammerstein equations with symmetric kernels having a finite number of negative eigenvalues.</i> (307) <i>Topological principles</i> (313)	
2. The principle of linearisation in problems on bifurcation points	316
<i>Functionals approximating to a quadratic.</i> (316) <i>L. A. Lyusternik's lemma.</i> (319) <i>The existence of a bifurcation point.</i> (322) <i>Auxiliary lemmas.</i> (325) <i>Uncontractible sets.</i> (329) <i>A basic theorem Example</i> (338)	
3. Existence theorems for eigenfunctions	342
<i>The Variational Principle.</i> (343) <i>M. Golomb's theorem.</i> (345) <i>A generalization of Golomb's theorem.</i> (347) <i>Kernels with a finite number of negative eigenvalues</i> (348)	
4. Stable critical points of even functionals	356
<i>L. A. Lyusternik's theorem.</i> (356) <i>The genus of a set.</i> (358) <i>The classes M_k.</i> (359) <i>The trace of a deformation.</i> (361) <i>Even functionals.</i> (362) <i>The separability of a space.</i> (363) <i>The critical numbers of a functional.</i> (364) <i>The critical points of a functional</i> (366) <i>The sets R_a.</i> (368) <i>The existence of different values of $d(a)$.</i> (370) <i>The norm of a functional.</i> (374) <i>The stability of a given critical value.</i> (375) <i>A fundamental theorem.</i> (378) <i>Remarks.</i> (379) <i>Odd Hammerstein operators.</i> (380) <i>Critical points on a hyperboloid.</i> (381) <i>Connection with problems on bifurcation points</i> (382)	
BIBLIOGRAPHY	383
SUBJECT INDEX	393

NOTE

As is clear from the above, the book is divided into six *chapters*, each chapter into *sections* (also referred to as § 1, § 2, etc.) and the sections into numbered *sub-sections* (the numbers of which do not appear in the table of contents). Lemmas, theorems and equations are numbered consecutively in each *section* (in § 2 the numbers are 2.1, 2.2, . . .); when reference is made to a lemma, theorem or equation from *another* section, the number of the section (or the page number) is specified. Some statements numbered with bold-face numbers are referred to as proposition I, etc. References (e.g. [44]) are to the Bibliography on p. 383.