

Contents

Preface	v
1 What Is Curvature?	1
The Euclidean Plane	1
Surfaces in Space	4
Curvature in Higher Dimensions	7
2 Riemannian Metrics	9
Definitions	9
Methods for Constructing Riemannian Metrics	15
Basic Constructions on Riemannian Manifolds	25
Lengths and Distances	33
Pseudo-Riemannian Metrics	40
Other Generalizations of Riemannian Metrics	46
Problems	47
3 Model Riemannian Manifolds	55
Symmetries of Riemannian Manifolds	55
Euclidean Spaces	57
Spheres	58
Hyperbolic Spaces	62
Invariant Metrics on Lie Groups	67
Other Homogeneous Riemannian Manifolds	72
Model Pseudo-Riemannian Manifolds	79
Problems	80
4 Connections	85
The Problem of Differentiating Vector Fields	85
Connections	88
Covariant Derivatives of Tensor Fields	95
Vector and Tensor Fields Along Curves	100

	Geodesics	103
	Parallel Transport	105
	Pullback Connections	110
	Problems	111
5	The Levi-Civita Connection	115
	The Tangential Connection Revisited	115
	Connections on Abstract Riemannian Manifolds	117
	The Exponential Map	126
	Normal Neighborhoods and Normal Coordinates	131
	Tubular Neighborhoods and Fermi Coordinates	133
	Geodesics of the Model Spaces	136
	Euclidean and Non-Euclidean Geometries	142
	Problems	145
6	Geodesics and Distance	151
	Geodesics and Minimizing Curves	151
	Uniformly Normal Neighborhoods	163
	Completeness	166
	Distance Functions	174
	Semigeodesic Coordinates	181
	Problems	185
7	Curvature	193
	Local Invariants	193
	The Curvature Tensor	196
	Flat Manifolds	199
	Symmetries of the Curvature Tensor	202
	The Ricci Identities	205
	Ricci and Scalar Curvatures	207
	The Weyl Tensor	212
	Curvatures of Conformally Related Metrics	216
	Problems	222
8	Riemannian Submanifolds	225
	The Second Fundamental Form	225
	Hypersurfaces	234
	Hypersurfaces in Euclidean Space	244
	Sectional Curvatures	250
	Problems	255
9	The Gauss–Bonnet Theorem	263
	Some Plane Geometry	263
	The Gauss–Bonnet Formula	271

The Gauss–Bonnet Theorem	276
Problems	281
10 Jacobi Fields	283
The Jacobi Equation	284
Basic Computations with Jacobi Fields	287
Conjugate Points.	297
The Second Variation Formula	300
Cut Points	307
Problems	313
11 Comparison Theory	319
Jacobi Fields, Hessians, and Riccati Equations	320
Comparisons Based on Sectional Curvature	327
Comparisons Based on Ricci Curvature	336
Problems	342
12 Curvature and Topology	345
Manifolds of Constant Curvature	345
Manifolds of Nonpositive Curvature	352
Manifolds of Positive Curvature	361
Problems	368
Appendix A: Review of Smooth Manifolds	371
Topological Preliminaries	371
Smooth Manifolds and Smooth Maps	374
Tangent Vectors	376
Submanifolds.	378
Vector Bundles	382
The Tangent Bundle and Vector Fields.	384
Smooth Covering Maps.	388
Appendix B: Review of Tensors.	391
Tensors on a Vector Space	391
Tensor Bundles and Tensor Fields	396
Differential Forms and Integration.	400
Densities	405
Appendix C: Review of Lie Groups.	407
Definitions and Properties	407
The Lie Algebra of a Lie Group	408
Group Actions on Manifolds.	411
References	415
Notation Index.	419
Subject Index.	423