

A SYNOPSIS
OF
ELEMENTARY RESULTS
IN
PURE AND APPLIED MATHEMATICS :

CONTAINING
PROPOSITIONS, FORMULÆ, AND METHODS OF ANALYSIS,
WITH
ABRIDGED DEMONSTRATIONS.

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