

Contents

1	Stable Solutions to Some Elliptic Problems: Minimal Cones, the Allen-Cahn Equation, and Blow-Up Solutions	1
	Xavier Cabré and Giorgio Poggesi	
1.1	Minimal Cones	2
1.1.1	The Simons Cone. Minimality	4
1.1.2	Hardy's Inequality	14
1.1.3	Proof of the Simons Lemma	16
1.1.4	Comments On: Harmonic Maps, Free Boundary Problems, and Nonlocal Minimal Surfaces	21
1.2	The Allen-Cahn Equation	25
1.2.1	Minimality of Monotone Solutions with Limits ± 1	27
1.2.2	A Conjecture of De Giorgi	31
1.2.3	The Saddle-Shaped Solution Vanishing on the Simons Cone	33
1.3	Blow-Up Problems	35
1.3.1	Stable and Extremal Solutions: A Singular Stable Solution for $n \geq 10$	36
1.3.2	Regularity of Stable Solutions. The Allard and Michael-Simon Sobolev Inequality	38
	References	44
2	Isoperimetric Inequalities for Eigenvalues of the Laplacian	47
	Antoine Henrot	
2.1	Notation and Prerequisites	48
2.1.1	Notation and Sobolev Spaces	48
2.1.2	Eigenvalues and Eigenfunctions	49
2.1.3	Properties of Eigenvalues	51
2.1.4	Some Examples	52
2.1.5	Min-Max Principles and Applications	53
2.1.6	Topological Derivative	55

2.2	The First Eigenvalue	56
2.2.1	The Faber-Krahn Inequality	56
2.2.2	A Quantitative Version of Faber-Krahn Inequality	57
2.2.3	The Case of Polygons	58
2.2.4	Domains in a Box	65
2.2.5	Multi-Connected Domains	69
2.3	The Second Eigenvalue	74
2.3.1	Minimizing λ_2	74
2.3.2	A Convexity Constraint.....	76
2.4	The Other Dirichlet Eigenvalues	78
2.4.1	Existence	78
2.4.2	Connectedness of Minimizers	79
2.4.3	Other Geometric Constraints	82
	References	86
3	Topological Aspects of Critical Points and Level Sets in Elliptic PDEs	89
	Alberto Enciso and Daniel Peralta-Salas	
3.1	Introduction: Emergence of Topological Structures in Elliptic PDEs	89
3.2	Critical Points of Green's Functions on Complete Manifolds	92
3.2.1	Li-Tam Green's Functions	93
3.2.2	A Topological Upper Bound on Surfaces.....	94
3.2.3	Critical Points in Higher Dimensions.....	99
3.3	General Strategy and Two Technical Tools: Thom's Isotopy Theorem and a Runge-Type Global Approximation Theorem.....	101
3.3.1	Thom's Isotopy Theorem.....	102
3.3.2	A Runge-Type Global Approximation Theorem for the Helmholtz Equation with Optimal Decay at Infinity.....	103
3.4	Monochromatic Waves: Nodal Sets of Solutions to the Helmholtz Equation	108
3.5	Emergence of Knotted Structures in High-Energy Eigenfunctions: Berry's Conjecture	110
3.6	The Linear Regime of Nonlinear Equations: Nodal Sets of the Allen-Cahn Equation	116
	References	118
4	Symmetry Properties for Solutions of Higher-Order Elliptic Boundary Value Problems	121
	Wolfgang Reichel	
4.1	Linear Problems: Weak Solutions, Eigenvalues, Regularity, Green Functions	122
4.2	Symmetry, Simplicity and Positivity for First Eigenfunctions	125
4.3	Symmetry for Nonlinear Problems by Uniqueness and Non-resonance.....	128

4.4	Symmetry via Moving Plane Method: An Example from Potential Theory	132
4.4.1	Asymptotic Expansion and the Role of the Barycenter	136
4.4.2	The Moving Plane Method	137
4.5	An Example of Symmetry in an Overdetermined 4th Order Problem	141
	References	144
5	Recent Trends in Free Boundary Regularity	147
	Henrik Shahgholian	
5.1	Introduction	147
5.1.1	Background	147
5.1.2	Overview of the Content	148
5.2	A Catalog of Semi-classical FBPs	149
5.2.1	A Melting Ice Block	149
5.2.2	Hele-Shaw Flow	149
5.2.3	An Optimal Stopping Problem	150
5.2.4	Modeling Financial Derivatives	151
5.2.5	Smash Sums and Internal DLA	152
5.3	Mathematical Theory of Obstacle Problem	153
5.3.1	Existence, and Uniqueness	153
5.3.2	Optimal Regularity and Non-degeneracy of Solutions.....	157
5.3.3	Regularity of FB: Local and Global Analysis	161
5.4	Other Types of FBPs	167
5.4.1	Bernoulli Type FB	167
5.4.2	Broken PDEs with FB	173
5.4.3	Non-variational Problems	178
5.4.4	Nonlocal Problems, Extensions and Thin Obstacles	182
5.5	System Case	187
5.5.1	Optimal Switching	187
5.5.2	Minimization Problems.....	189
	References	196