

Theories, Sites, Toposes

*Relating and studying mathematical theories
through topos-theoretic ‘bridges’*

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