Contents

List of Symbols

Part I **Theory and Techniques** 1

Chapter 1 **Historical Perspective** 3

1.1 The Nature of Systems with Periodically Time-Varying Parameters .. 3
1.2 1831–1887 Faraday to Rayleigh–Early Experimentalists and Theorists 7
1.3 1918–1940 The First Applications 9
1.4 Second Generation Applications 10
1.5 Recent Theoretical Developments 12
1.6 Commonplace Illustrations of Parametric Behaviour 12
References for Chapter 1 14
Problems 16

Chapter 2 **The Equations and Their Properties** 17

2.1 Hill Equations 17
2.2 Matrix Formulation of Hill Equations 18
2.3 The State Transition Matrix 19
2.4 Floquet Theory 20
2.5 Second Order Systems 22
2.6 Natural Modes of Solution 23
2.7 Concluding Comments 24
References for Chapter 2 25
Problems 25

Chapter 3 **Solutions to Periodic Differential Equations** 27

3.1 Solutions Over One Period of the Coefficient 27
3.2 The Meissner Equation 28
3.3 Solution at Any Time for a Second Order Periodic Equation 29
3.4 Evaluation of $\phi(\pi, 0)^m$, m Integral 30
3.5 The Hill Equation with a Staircase Coefficient 32
3.6 The Hill Equation with a Sawtooth Waveform Coefficient 32
3.6.1 The Wronskian Matrix with z Negative 35
3.6.2 The Wronskian Matrix with z Zero 35
3.6.3 The Case of β Negative 36
3.7 The Hill Equation with a Positive Slope, Sawtooth Waveform Coefficient 36
3.8 The Hill Equation with a Triangular Coefficient 37

3.9 The Hill Equation with a Trapezoidal Coefficient 38
3.10 Bessel Function Generation 38
3.11 The Hill Equation with a Repetitive Exponential Coefficient 39
3.12 The Hill Equation with a Coefficient in the Form of a Repetitive Sequence of Impulses 40
3.13 Equations of Higher Order 41
3.14 Response to a Sinusoidal Forcing Function 41
3.15 Phase Space Analysis 44
3.16 Concluding Comments 46
References for Chapter 3 48
Problems 49

Chapter 4 **Stability** 50

4.1 Types of Stability 50
4.2 Stability Theorems for Periodic Systems 51
4.3 Second Order Systems 52
4.3.1 Stability and the Characteristic Exponent 52
4.3.2 The Meissner Equation 53
4.3.3 The Hill Equation with an Impulsive Coefficient 56
4.3.4 The Hill Equation with a Sawtooth Waveform Coefficient 57
4.3.5 The Hill Equation with a Triangular Waveform Coefficient 57
4.3.6 Hill Determinant Analysis 57
4.3.7 Parametric Frequencies for Second Order Systems 62
4.4 General Order Systems 63
4.4.1 Hill Determinant Analysis for General Order Systems 63
4.4.2 Residues of the Hill Determinant for $q \to 0$ 66
4.4.3 Instability and Parametric Frequencies for General Systems 67
4.4.4 Stability Diagrams for General Order Systems 67
4.5 Natural Modes and Mode Diagrams 68
4.5.1 Nature of the Basis Solutions 68
4.5.2 *P* Type Solutions 69
4.5.3 *C* Type Solutions 70
4.5.4 *N* Type Solutions 70
4.5.5 Modes of Solution 71
4.5.6 The Modes of a Second Order Periodic System 71
4.5.7 Boundary Modes 72
4.5.8 Second Order System with Losses 73
4.5.9 Modes for Systems of General Order 73
4.5.10 Coexistence 74
4.6 Short Time Stability 75
References for Chapter 4 79
Problems 79

Chapter 5 **A Modelling Technique for Hill Equations** 81

5.1 Convergence of the Hill Determinant and Significance of the Harmonics of the Periodic Coefficients 81
5.1.1 Second Order Systems 81
5.1.2 General Order Systems 84
5.2 A Modelling Philosophy for Intractable Hill Equations 84
5.3 The Frequency Spectrum of a Periodic Staircase Coefficient 85
5.4 Piecewise Linear Models 87
5.4.1 General Comments 87
5.4.2 Trapezoidal Models 87

5.5 Forced Response Modelling ... 88
5.6 Stability Diagram and Characteristic Exponent Modelling ... 88
5.7 Models for Nonlinear Hill Equations ... 88
5.8 A Note on Discrete Spectral Analysis ... 89
5.9 Concluding Remarks ... 90
References for Chapter 5 ... 91
Problems ... 91

Chapter 6 **The Mathieu Equation** ... 93

6.1 Classical Methods for Analysis and Their Limitations ... 93
6.1.1 Periodic Solutions ... 93
6.1.2 Mathieu Functions of Fractional Order ... 95
6.1.3 Fractional Order Unstable Solutions ... 96
6.1.4 Limitations of the Classical Method of Treatment ... 96
6.2 Numerical Solution of the Mathieu Equation ... 98
6.3 Modelling Techniques for Analysis ... 99
6.3.1 Rectangular Waveform Models ... 99
6.3.2 Trapezoidal Waveform Models ... 100
6.3.3 Staircase Waveform Models ... 101
6.3.4 Performance Comparison of the Models ... 102
6.4 Stability Diagrams for the Mathieu Equation ... 103
6.4.1 The Lossless Mathieu Equation ... 103
6.4.2 The Damped (Lossy) Mathieu Equation ... 105
6.4.3 Sufficient Conditions for the Stability of the Damped Mathieu Equation ... 106
References for Chapter 6 ... 106
Problems ... 107

Part II Applications ... 109

Chapter 7 **Practical Periodically Variable Systems** ... 111

7.1 The Quadrupole Mass Spectrometer ... 111
7.1.1 Spatially Linear Electric Fields ... 112
7.1.2 The Quadrupole Mass Filter ... 113
7.1.3 The Monopole Mass Spectrometer ... 117
7.1.4 The Quadrupole Ion Trap ... 120
7.1.5 Simulation of Quadrupole Devices ... 120
7.1.6 Non idealities in Quadrupole Devices ... 123
7.2 Dynamic Buckling of Structures ... 123
7.3 Elliptical Waveguides ... 127
7.3.1 The Helmholtz Equation ... 128
7.3.2 Rectangular Waveguides ... 129
7.3.3 Circular Waveguides ... 131
7.3.4 Elliptical Waveguides ... 133
7.3.5 Computation of the Cut-off Frequencies for an Elliptical Waveguide ... 136
7.4 Wave Propagation in Periodic Media ... 137
7.4.1 Pass and Stop Bands ... 138
7.4.2 The $\omega - \beta_r$ (Brillouin) Diagram ... 140
7.4.3 Electromagnetic Wave Propagation in Periodic Media ... 143
7.4.4 Guided Electromagnetic Wave Propagation in Periodic Media ... 144
7.4.5 Electrons in Crystal Lattices ... 145
7.4.6 Other Examples of Waves in Periodic Media ... 149

7.5	Electric Circuit Applications	150
7.5.1	Degenerate Parametric Amplification	151
7.5.2	Degenerate Parametric Amplification in High Order Periodic Networks	154
7.5.3	Nondegenerate Parametric Amplification	154
7.5.4	Parametric Up Converters	155
7.5.5	*N*-path Networks	158
	References for Chapter 7	162
	Problems	165
Appendix	**Bessel Function Generation by Chebyshev Polynomial Methods**	168
	References for Appendix	169
Subject Index		171