

Cédric Bonnafé

# Representations of $SL_2(\mathbb{F}_q)$

 Springer

# Contents

## Part I Preliminaries

|          |   |           |
|----------|---|-----------|
| <b>1</b> | <b>Structure of <math>SL_2(\mathbb{F}_q)</math></b> ..... | <b>3</b>  |
| 1.1      | Special Subgroups .....                                   | 3         |
| 1.1.1    | Bruhat Decomposition .....                                | 3         |
| 1.1.2    | The Non-Split Torus .....                                 | 5         |
| 1.2      | Distinguished Subgroups .....                             | 6         |
| 1.3      | Conjugacy Classes .....                                   | 7         |
| 1.3.1    | Centralisers .....  | 7         |
| 1.3.2    | Parametrisation .....                                     | 8         |
| 1.4      | Sylow Subgroups .....                                     | 9         |
| 1.4.1    | Sylow $p$ -Subgroups .....                                | 10        |
| 1.4.2    | Other Sylow Subgroups .....                               | 10        |
| <b>2</b> | <b>The Geometry of the Drinfeld Curve</b> .....           | <b>15</b> |
| 2.1      | Elementary Properties .....                               | 16        |
| 2.2      | Interesting Quotients .....                               | 16        |
| 2.2.1    | Quotient by $G$ .....                                     | 17        |
| 2.2.2    | Quotient by $U$ .....                                     | 18        |
| 2.2.3    | Quotient by $\mu_{q+1}$ .....                             | 19        |
| 2.3      | Fixed Points under certain Frobenius Endomorphisms .....  | 19        |
| 2.4      | Compactification .....                                    | 20        |
| 2.5      | Curiosities* .....  | 21        |
| 2.5.1    | Hurwitz Formula, Automorphisms* .....                     | 22        |
| 2.5.2    | Abhyankar's Conjecture (Raynaud's Theorem)* .....         | 24        |

## Part II Ordinary Characters

|          |                                       |           |
|----------|---------------------------------------|-----------|
| <b>3</b> | <b>Harish-Chandra Induction</b> ..... | <b>29</b> |
| 3.1      | Bimodules .....                       | 29        |
| 3.2      | Harish-Chandra Induction .....        | 30        |

|   |   |           |
|---|---|-----------|
| 3.2.1                                   | Definition.....   | 30        |
| 3.2.2                                   | Other Constructions .....   | 31        |
| 3.2.3                                   | Mackey Formula .....  | 32        |
| 3.2.4                                   | Restriction from $GL_2(\mathbb{F}_q)$ .....   | 33        |
| 3.2.5                                   | Summary .....   | 34        |
| <b>4</b>                                | <b>Deligne-Lusztig Induction.....</b>   | <b>37</b> |
| 4.1                                     | Definition and First Properties .....   | 37        |
| 4.1.1                                   | Definition.....   | 37        |
| 4.1.2                                   | The Character $R'(1)$ .....   | 38        |
| 4.1.3                                   | Dimensions .....  | 39        |
| 4.1.4                                   | Cuspidality .....   | 39        |
| 4.2                                     | Mackey Formula.....   | 40        |
| 4.3                                     | Parametrisation of $\text{Irr } G$ .....  | 45        |
| 4.4                                     | Action of the Frobenius Endomorphism .....  | 46        |
| 4.4.1                                   | Action on $H_c^1(\mathbf{Y})e_1$ .....  | 46        |
| 4.4.2                                   | Action on $H_c^1(\mathbf{Y})e_{\theta_0}$ .....                                       | 47        |
| 4.4.3                                   | Action on $H_c^1(\mathbf{Y})e_{\theta} \oplus H_c^1(\mathbf{Y})e_{\theta^{-1}}$ ..... | 48        |
| <b>5</b>                                | <b>The Character Table .....</b>  | <b>51</b> |
| 5.1                                     | Characters of Bimodules .....   | 51        |
| 5.1.1                                   | Calculation of $\mathfrak{T}$ .....   | 51        |
| 5.1.2                                   | Calculation of $\mathfrak{T}'$ .....  | 52        |
| 5.1.3                                   | The Characters $R(\alpha)$ and $R'(\theta)$ .....                                     | 53        |
| 5.2                                     | Restriction to $U$ .....  | 53        |
| 5.2.1                                   | $B$ -Invariant Characters of $U$ .....  | 54        |
| 5.2.2                                   | Restriction of Characters of $G$ .....  | 54        |
| 5.2.3                                   | Values of $\Upsilon_{\pm}$ .....  | 56        |
| 5.3                                     | Character Table .....   | 57        |
| <b>Part III Modular Representations</b> |   |           |
| <b>6</b>                                | <b>More about Characters of <math>G</math> and of its Sylow Subgroups .....</b>       | <b>63</b> |
| 6.1                                     | Central Characters .....  | 63        |
| 6.2                                     | Global McKay Conjecture .....   | 64        |
| 6.2.1                                   | Characters of $N$ .....   | 65        |
| 6.2.2                                   | Characters of $N'$ .....  | 66        |
| 6.2.3                                   | Characters of $B$ .....   | 67        |
| 6.2.4                                   | Normalisers of Sylow 2-Subgroups .....  | 68        |
| 6.2.5                                   | Verification of the Global McKay Conjecture .....                                     | 68        |
| <b>7</b>                                | <b>Unequal Characteristic: Generalities .....</b>                                     | <b>71</b> |
| 7.1                                     | Blocks, Brauer Correspondents.....  | 71        |
| 7.1.1                                   | Partition in $\ell$ -Blocks .....   | 71        |
| 7.1.2                                   | Brauer Correspondents .....   | 72        |
| 7.1.3                                   | Terminology .....   | 74        |

|                            |   |            |
|----------------------------|---|------------|
| 7.2                        | Modular Harish-Chandra Induction .....  | 75         |
| 7.3                        | Deligne-Lusztig Induction* .....  | 77         |
| <b>8</b>                   | <b>Unequal Characteristic: Equivalences of Categories .....</b>                 | <b>85</b>  |
| 8.1                        | Nilpotent Blocks .....  | 86         |
| 8.1.1                      | Harish-Chandra Induction .....  | 86         |
| 8.1.2                      | Deligne-Lusztig Induction* .....  | 87         |
| 8.2                        | Quasi-Isolated Blocks .....   | 88         |
| 8.2.1                      | Harish-Chandra Induction .....  | 88         |
| 8.2.2                      | Deligne-Lusztig Induction* .....  | 88         |
| 8.3                        | The Principal Block .....   | 89         |
| 8.3.1                      | The Case when $\ell$ is Odd and Divides $q - 1$ .....                           | 89         |
| 8.3.2                      | The Case when $\ell$ is Odd and Divides $q + 1^*$ .....                         | 90         |
| 8.3.3                      | The Case when $\ell = 2^*$ .....  | 93         |
| 8.4                        | Alvis-Curtis Duality* .....   | 94         |
| <b>9</b>                   | <b>Unequal Characteristic: Simple Modules, Decomposition<br/>Matrices .....</b> | <b>97</b>  |
| 9.1                        | Preliminaries .....   | 97         |
| 9.1.1                      | Induction and Decomposition Matrices .....                                      | 97         |
| 9.1.2                      | Dimensions of Modules and Restriction to $U$ .....                              | 98         |
| 9.2                        | Nilpotent Blocks .....  | 99         |
| 9.3                        | Quasi-Isolated Blocks .....   | 100        |
| 9.4                        | The Principal Block .....   | 101        |
| 9.4.1                      | Preliminaries .....   | 101        |
| 9.4.2                      | The Case when $\ell$ is Odd and Divides $q - 1$ .....                           | 103        |
| 9.4.3                      | The Case when $\ell$ is Odd and Divides $q + 1$ .....                           | 104        |
| 9.4.4                      | The Case when $\ell = 2$ .....  | 105        |
| <b>10</b>                  | <b>Equal Characteristic .....</b>   | <b>109</b> |
| 10.1                       | Simple Modules .....  | 110        |
| 10.1.1                     | Standard or Weyl Modules .....  | 110        |
| 10.1.2                     | Simple Modules .....  | 113        |
| 10.1.3                     | The Grothendieck Ring of $\mathbf{G}$ .....                                     | 117        |
| 10.2                       | Simple $kG$ -Modules and Decomposition Matrices .....                           | 119        |
| 10.2.1                     | Simple $kG$ -Modules .....  | 119        |
| 10.2.2                     | Decomposition Matrices .....  | 121        |
| 10.3                       | Blocks .....  | 122        |
| 10.3.1                     | Blocks and Brauer Correspondents .....  | 122        |
| 10.3.2                     | Brauer Trees* .....   | 123        |
| <b>Part IV Complements</b> |   |            |
| <b>11</b>                  | <b>Special Cases .....</b>  | <b>129</b> |
| 11.1                       | Preliminaries .....   | 129        |
| 11.2                       | The Case when $q = 3$ .....   | 131        |

|                   |  |            |
|-------------------|--|------------|
| 11.2.1            | Structure .....  | 131        |
| 11.2.2            | Character Table .....  | 132        |
| 11.2.3            | The Group $SL_2(\mathbb{F}_3)$ as a Subgroup of $SL_2(\mathbb{F}_\ell)$ .....                      | 133        |
| 11.2.4            | The Group $SL_2(\mathbb{F}_3)$ as a Reflection Group of Rank 2 ...                                 | 134        |
| 11.2.5            | The Group $PSL_2(\mathbb{F}_3)$ and the Isometries of the Tetrahedron.....                         | 135        |
| 11.3              | The Case when $q = 5$ .....  | 136        |
| 11.3.1            | Structure .....  | 136        |
| 11.3.2            | Character Table .....  | 136        |
| 11.3.3            | The Group $SL_2(\mathbb{F}_5)$ as a Subgroup of $SL_2(\mathbb{F}_{\ell^r})$ .....                  | 137        |
| 11.3.4            | The Group $SL_2(\mathbb{F}_5) \times \mathbb{Z}/5\mathbb{Z}$ as a Reflection Group of Rank 2.....  | 139        |
| 11.3.5            | The Group $PSL_2(\mathbb{F}_5)$ , the Dodecahedron and the Icosahedron.....                        | 141        |
| 11.4              | The Case when $q = 7$ .....  | 142        |
| 11.4.1            | Structure .....  | 143        |
| 11.4.2            | Character Table .....  | 143        |
| 11.4.3            | The Isomorphism Between the Groups $PSL_2(\mathbb{F}_7)$ and $GL_3(\mathbb{F}_2)$ .....            | 143        |
| 11.4.4            | The Group $PSL_2(\mathbb{F}_7) \times \mathbb{Z}/2\mathbb{Z}$ as a Reflection Group of Rank 3..... | 145        |
| <b>12</b>         | <b>Deligne-Lusztig Theory: an Overview*</b> .....  | <b>149</b> |
| 12.1              | Deligne-Lusztig Induction .....  | 150        |
| 12.2              | Modular Representations .....  | 154        |
| 12.2.1            | Blocks .....   | 154        |
| 12.2.2            | Modular Deligne-Lusztig Induction.....   | 155        |
| 12.2.3            | The Geometric Version of Broué's Conjecture .....  | 155        |
| <b>Appendix A</b> | <b><math>\ell</math>-Adic Cohomology</b> .....   | <b>159</b> |
| A.1               | Properties of the Complex* .....   | 159        |
| A.2               | Properties of the Cohomology Groups .....  | 160        |
| A.2.1             | General Properties.....  | 160        |
| A.2.2             | Cohomology with Coefficients in $K$ .....  | 161        |
| A.2.3             | The Euler Characteristic .....   | 161        |
| A.2.4             | Action of a Frobenius Endomorphism.....  | 162        |
| A.3               | Examples .....   | 163        |
| A.3.1             | The Projective Line .....  | 163        |
| A.3.2             | The One-Dimensional Torus.....   | 164        |
| <b>Appendix B</b> | <b>Block Theory</b> .....  | <b>167</b> |
| B.1               | Definition .....   | 167        |
| B.2               | Brauer Correspondents.....   | 168        |
| B.2.1             | Brauer's Theorems .....  | 168        |
| B.2.2             | Conjectures .....  | 170        |

|   |            |
|---|------------|
| Contents  | xvii       |
| B.2.3 Equivalences of Categories: Methods .....     | 170        |
| B.3 Decomposition Matrices .....                    | 173        |
| B.4 Brauer Trees* .....                             | 174        |
| <b>Appendix C Review of Reflection Groups .....</b> | <b>177</b> |
| <b>References .....</b>                             | <b>179</b> |
| <b>Index .....</b>                                  | <b>183</b> |