Contents

Preface vii

Introduction 1

Notation and conventions 13

1 Preliminaries on Lie groups 15
- 1.1 Lie groups, representations, and Fourier transform 15
- 1.2 Lie algebras and vector fields 22
- 1.3 Universal enveloping algebra and differential operators 24
- 1.4 Distributions and Schwartz kernel theorem 29
- 1.5 Convolutions 30
- 1.6 Nilpotent Lie groups and algebras 34
- 1.7 Smooth vectors and infinitesimal representations 37
- 1.8 Plancherel theorem 43
  - 1.8.1 Orbit method 44
  - 1.8.2 Plancherel theorem and group von Neumann algebras . . . 46
  - 1.8.3 Fields of operators acting on smooth vectors 52

2 Quantization on compact Lie groups 57
- 2.1 Fourier analysis on compact Lie groups 58
  - 2.1.1 Characters and tensor products 58
  - 2.1.2 Peter-Weyl theorem 60
  - 2.1.3 Spaces of functions and distributions on G 63
  - 2.1.4 ℓ^p-spaces on the unitary dual $\widehat{G}$ 67
- 2.2 Pseudo-differential operators on compact Lie groups 71
  - 2.2.1 Symbols and quantization 71
  - 2.2.2 Difference operators and symbol classes 74
  - 2.2.3 Symbolic calculus, ellipticity, hypoellipticity 78
  - 2.2.4 Fourier multipliers and L^p-boundedness 82
  - 2.2.5 Sharp Gårding inequality 88

3 Homogeneous Lie groups **91**
3.1 Graded and homogeneous Lie groups 92
3.1.1 Definition and examples of graded Lie groups 92
3.1.2 Definition and examples of homogeneous Lie groups 94
3.1.3 Homogeneous structure 100
3.1.4 Polynomials 103
3.1.5 Invariant differential operators on homogeneous Lie groups 105
3.1.6 Homogeneous quasi-norms 109
3.1.7 Polar coordinates 116
3.1.8 Mean value theorem and Taylor expansion 119
3.1.9 Schwartz space and tempered distributions 125
3.1.10 Approximation of the identity 129
3.2 Operators on homogeneous Lie groups 132
3.2.1 Left-invariant operators on homogeneous Lie groups 132
3.2.2 Left-invariant homogeneous operators 136
3.2.3 Singular integral operators on homogeneous Lie groups . . . 140
3.2.4 Principal value distribution 146
3.2.5 Operators of type $\nu = 0$ 151
3.2.6 Properties of kernels of type ν, $\operatorname{Re}\nu \in [0, Q)$ 154
3.2.7 Fundamental solutions of homogeneous differential operators 158
3.2.8 Liouville's theorem on homogeneous Lie groups 167

4 Rockland operators and Sobolev spaces **171**
4.1 Rockland operators 172
4.1.1 Definition of Rockland operators 172
4.1.2 Examples of Rockland operators 174
4.1.3 Hypoellipticity and functional calculus 177
4.2 Positive Rockland operators 183
4.2.1 First properties 183
4.2.2 The heat semi-group and the heat kernel 185
4.2.3 Proof of the heat kernel theorem and its corollaries 187
4.3 Fractional powers of positive Rockland operators 198
4.3.1 Positive Rockland operators on L^p 198
4.3.2 Fractional powers of operators $\mathcal{R}_p$ 203
4.3.3 Imaginary powers of $\mathcal{R}_p$ and $\mathrm{I} + \mathcal{R}_p$ 206
4.3.4 Riesz and Bessel potentials 211
4.4 Sobolev spaces on graded Lie groups 218
4.4.1 (Inhomogeneous) Sobolev spaces 218
4.4.2 Interpolation between inhomogeneous Sobolev spaces 225
4.4.3 Homogeneous Sobolev spaces 228
4.4.4 Operators acting on Sobolev spaces 233
4.4.5 Independence in Rockland operators and integer orders . . 236
4.4.6 Sobolev embeddings 239
4.4.7 List of properties for the Sobolev spaces 245

4.4.8 Right invariant Rockland operators and Sobolev spaces . . 249
4.5 Hulanicki's theorem 251
4.5.1 Statement 251
4.5.2 Proof of Hulanicki's theorem 252
4.5.3 Proof of Corollary 4.5.2 269

5 Quantization on graded Lie groups **271**
5.1 Symbols and quantization 272
5.1.1 Fourier transform on Sobolev spaces 273
5.1.2 The spaces $\mathcal{K}_{a,b}(G)$, $\mathscr{L}_L(L^2_a(G), L^2_b(G))$, and $L^\infty_{a,b}(\widehat{G})$ 285
5.1.3 Symbols and associated kernels 294
5.1.4 Quantization formula 296
5.2 Symbol classes $S^m_{\rho,\delta}$ and operator classes $\Psi^m_{\rho,\delta}$ 300
5.2.1 Difference operators 300
5.2.2 Symbol classes $S^m_{\rho,\delta}$ 306
5.2.3 Operator classes $\Psi^m_{\rho,\delta}$ 309
5.2.4 First examples 313
5.2.5 First properties of symbol classes 316
5.3 Spectral multipliers in positive Rockland operators 319
5.3.1 Multipliers in one positive Rockland operator 319
5.3.2 Joint multipliers 327
5.4 Kernels of pseudo-differential operators 330
5.4.1 Estimates of the kernels 330
5.4.2 Smoothing operators and symbols 339
5.4.3 Pseudo-differential operators as limits of smoothing operators 341
5.4.4 Operators in Ψ^0 as singular integral operators 345
5.5 Symbolic calculus 351
5.5.1 Asymptotic sums of symbols 351
5.5.2 Composition of pseudo-differential operators 353
5.5.3 Adjoint of a pseudo-differential operator 364
5.5.4 Simplification of the definition of $S^m_{\rho,\delta}$ 371
5.6 Amplitudes and amplitude operators 374
5.6.1 Definition and quantization 374
5.6.2 Amplitude classes 379
5.6.3 Properties of amplitude classes and kernels 381
5.6.4 Link between symbols and amplitudes 384
5.7 Calderón-Vaillancourt theorem 385
5.7.1 Analogue of the decomposition into unit cubes 387
5.7.2 Proof of the case $S^0_{0,0}$ 389
5.7.3 A bilinear estimate 396
5.7.4 Proof of the case $S^0_{\rho,\rho}$ 403
5.8 Parametrices, ellipticity and hypoellipticity 409
5.8.1 Ellipticity 410
5.8.2 Parametrix 417

5.8.3 Subelliptic estimates and hypoellipticity 423

6 Pseudo-differential operators on the Heisenberg group 427
6.1 Preliminaries 428
6.1.1 Descriptions of the Heisenberg group 428
6.1.2 Heisenberg Lie algebra and the stratified structure 430
6.2 Dual of the Heisenberg group 431
6.2.1 Schrödinger representations π_λ 432
6.2.2 Group Fourier transform on the Heisenberg group 433
6.2.3 Plancherel measure 441
6.3 Difference operators 443
6.3.1 Difference operators Δ_{x_j} and Δ_{y_j} 443
6.3.2 Difference operator Δ_t 447
6.3.3 Formulae 453
6.4 Shubin classes 455
6.4.1 Weyl-Hörmander calculus 455
6.4.2 Shubin classes $\Sigma^m_\rho(\mathbb{R}^n)$ and the harmonic oscillator 459
6.4.3 Shubin Sobolev spaces 461
6.4.4 The λ-Shubin classes $\Sigma^m_{\rho,\lambda}(\mathbb{R}^n)$ 468
6.4.5 Commutator characterisation of λ-Shubin classes 473
6.5 Quantization and symbol classes $S^m_{\rho,\delta}$ on the Heisenberg group . . 475
6.5.1 Quantization on the Heisenberg group 475
6.5.2 An equivalent family of seminorms on $S^m_{\rho,\delta} = S^m_{\rho,\delta}(\mathbb{H}_n)$ . . . 477
6.5.3 Characterisation of $S^m_{\rho,\delta}(\mathbb{H}_n)$ 478
6.6 Parametrices 481
6.6.1 Condition for ellipticity 481
6.6.2 Condition for hypoellipticity 483
6.6.3 Subelliptic estimates and hypoellipticity 486

A Miscellaneous 491
A.1 General properties of hypoelliptic operators 491
A.2 Semi-groups of operators 493
A.3 Fractional powers of operators 495
A.4 Singular integrals (according to Coifman-Weiss) 499
A.5 Almost orthogonality 503
A.6 Interpolation of analytic families of operators 506

B Group C^* and von Neumann algebras 509
B.1 Direct integral of Hilbert spaces 509
B.1.1 Convention: Hilbert spaces are assumed separable 509
B.1.2 Measurable fields of vectors 510
B.1.3 Direct integral of tensor products of Hilbert spaces 511
B.1.4 Separability of a direct integral of Hilbert spaces 514
B.1.5 Measurable fields of operators 515

B.1.6 Integral of representations 516
B.2 C^*- and von Neumann algebras 517
B.2.1 Generalities on algebras 517
B.2.2 C^*-algebras 520
B.2.3 Group C^*-algebras 521
B.2.4 Von Neumann algebras 524
B.2.5 Group von Neumann algebra 525
B.2.6 Decomposition of group von Neumann algebras and abstract Plancherel theorem 527

Schrödinger representations and Weyl quantization **531**

Explicit symbolic calculus on the Heisenberg group **532**

List of quantizations **533**

Bibliography **535**

Index **553**