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Averaging for Nonlinear Dynamics with Applications and Numerical Bifurcations

Parametric and autoparametric systems,
Hamiltonian systems, FPU systems, coupled
oscillators and chaos

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