
Contents

List of figures	xi
Preface	xiii
List of abbreviations	xv
Introduction	1
1 Geometric background	15
1.1 Analytic geometry	15
1.1.1 Analytic varieties	15
1.1.2 The non-Archimedean affine and projective lines	16
1.1.3 Non-Archimedean Berkovich curves	17
1.2 Potential theory	20
1.2.1 Pluripotential theory on complex manifolds	20
1.2.2 Potential theory on Berkovich analytic curves	22
1.2.3 Subharmonic functions on singular curves	26
1.3 Line bundles on curves	27
1.3.1 Metrizations of line bundles	27
1.3.2 Positive line bundles	28
1.4 Adelic metrics, Arakelov heights, and equidistribution	29
1.4.1 Number fields	29
1.4.2 Adelic metrics	30
1.4.3 Heights	30
1.4.4 Equidistribution	31
1.5 Adelic series and Xie's algebraization theorem	32
2 Polynomial dynamics	37
2.1 The parameter space of polynomials	37
2.2 Fatou-Julia theory	39
2.3 Green functions and equilibrium measure	43
2.3.1 Basic definitions	43
2.3.2 Estimates on the Green function	44
2.4 Examples	47
2.4.1 Integrable polynomials	47
2.4.2 Potential good reduction	48
2.4.3 PCF maps	49
2.5 Böttcher coordinates and Green functions	49

2.5.1	Expansion of the Böttcher coordinate	49
2.5.2	Böttcher coordinate and Green function	53
2.6	Polynomial dynamics over a global field	56
2.7	Bifurcations in holomorphic dynamics	58
2.8	Components of preperiodic points	60
3	Dynamical symmetries	64
3.1	The group of dynamical symmetries of a polynomial	64
3.2	Symmetry groups in family	68
3.3	Algebraic characterization of dynamical symmetries	69
3.4	Primitive families of polynomials	72
3.5	Ritt's theory of composite polynomials	76
3.5.1	Decomposability	77
3.5.2	Intertwined polynomials	78
3.5.3	Uniform bounds and invariant subvarieties	81
3.5.4	Intertwining classes	81
3.5.5	Intertwining classes of a generic polynomial	83
3.6	Stratification of the parameter space in low degree	85
3.7	Open problems	87
4	Polynomial dynamical pairs	89
4.1	Holomorphic dynamical pairs and proof of Theorem 4.10	89
4.1.1	Basics on holomorphic dynamical pairs	90
4.1.2	Density of transversely prerepelling parameters	91
4.1.3	Rigidity of the bifurcation locus	94
4.1.4	A renormalization procedure	96
4.1.5	Bifurcation locus of a dynamical pair and J -stability	98
4.1.6	Proof of Theorem 4.10	98
4.2	Algebraic dynamical pairs	102
4.2.1	Algebraic dynamical pairs	102
4.2.2	The divisor of a dynamical pair	102
4.2.3	Meromorphic dynamical pairs parametrized by the punctured disk	103
4.2.4	Metritizations and dynamical pairs	109
4.2.5	Characterization of passivity	111
4.3	Family of polynomials and Green functions	113
4.4	Arithmetic polynomial dynamical pairs	114
5	Entanglement of dynamical pairs	119
5.1	Dynamical entanglement	119
5.1.1	Definition	119
5.1.2	Characterization of entanglement	120
5.1.3	Overview of the proof of Theorem B	121
5.2	Dynamical pairs with identical measures	122
5.2.1	Equality at an Archimedean place	122

5.2.2	The implication $(1) \Rightarrow (2)$ of Theorem B	126
5.3	Multiplicative dependence of the degrees	127
5.4	Proof of the implication $(2) \Rightarrow (3)$ of Theorem B	131
5.4.1	More precise forms of Theorem B	139
5.5	Proof of Theorem C	140
5.6	Further results and open problems	145
5.6.1	Effective versions of the theorem	145
5.6.2	The integrable case	146
5.6.3	Algorithm	146
5.6.4	Application to Manin-Mumford's problem	147
6	Entanglement of marked points	150
6.1	Proof of Theorem D	150
6.2	Proof of Theorem E	152
7	The unicritical family	157
7.1	General facts	157
7.2	Unlikely intersection in the unicritical family	160
7.3	Archimedean rigidity	161
7.4	Connectedness of the bifurcation locus	162
7.5	Some experiments	163
8	Special curves	169
8.1	Special curves in the moduli space of polynomials	170
8.2	Marked dynamical graphs	171
8.2.1	Definition	171
8.2.2	Critically marked dynamical graphs	173
8.2.3	The critical graph of a polynomial	174
8.2.4	The critical graph of an irreducible subvariety in the moduli space of polynomials	175
8.3	Dynamical graphs attached to special curves	176
8.4	Realization theorem	177
8.4.1	Asymmetric special graphs	179
8.4.2	Truncated marked dynamical graphs	180
8.4.3	Polynomials with a fixed portrait	181
8.4.4	Construction of a suitable sequence of Riemann surfaces	185
8.4.5	End of the proof of Theorem 8.15	197
8.5	Special curves and dynamical graphs	198
8.5.1	Wringing deformations and marked dynamical graphs	199
8.5.2	Proof of Theorem 8.30	201
8.6	Realizability of PCF maps	202
8.6.1	Proof of Proposition 8.34	203
8.6.2	Combinatorics of strictly PCF polynomials	205
8.6.3	Proof of Proposition 8.35	208
8.7	Special curves in low degrees	210

8.8	Open questions on the geometry of special curves	212
	Notes	215
	Bibliography	217
	Index	231

List of figures

4.1	A real bifurcation locus	95
4.2	Bifurcation loci from Example 4.24	105
7.1	The Multibrot set $\mathcal{M}(3,0)$, and the locus $\mathcal{M}(3,-it)$	166
7.2	Lotus shapes in the froggy bifurcation locus	167
7.3	An approximation of \mathbb{M}	168
8.1	Flows on graphs	172
8.2	Two examples of non-asymmetric graphs	174
8.3	Special critically marked dynamical graphs of degree 2	178
8.4	Special critically marked dynamical graphs of degree 3	178
8.5	Example where $\mathcal{Q}(p) \subset \gamma'$ but $\mathcal{D}_0(p) \subsetneq \mathcal{D}(p)$	182
8.6	Non-uniqueness of the realization of a critical portrait	183
8.7	Constructing the surfaces S_n	187
8.8	The patching procedure when $A \neq A_\star$	189
8.9	The patching procedure when $A = A_\star$	190
8.10	A tree of circles	192
8.11	A critical portrait	207
8.12	Asymmetrical special critically marked dynamical graphs of degree 4	211