A Hierarchy of Turing Degrees

A Transfinite Hierarchy of Lowness Notions in the Computably Enumerable Degrees, Unifying Classes, and Natural Definability

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PRINCETON UNIVERSITY PRESS
PRINCETON AND OXFORD
2020

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